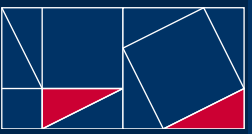

INSTITUTE FOR MATHEMATICS & EDUCATION

Finding the Thread: Tracing a Mathematical Idea from Elementary to High School

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September 2008

Dividing Fractions

$$\frac{7/8}{3/4} = \frac{7}{8} \cdot \frac{4}{3}$$

The Thread

The Idea of Equivalence

Different forms with the same value

The Idea of Equivalence (alternative version)

Different views of the same thing

Max finds a theorem (from DMI, Schifter et al)

A group of fourth graders is discussing a subtraction problem:

$$145 - 98 = 47.$$

They are trying to figure out how to use $145 - 100 = 45$.

Max's insight

Yeah, the less you subtract, the more you end up with. AND . . . in fact the thing you end up with is exactly as much larger as the amount less that you subtracted.

What equivalence of forms is Max describing?

$$145 - (100 - 2) \qquad (145 - 100) + 2$$

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Finding the Thread

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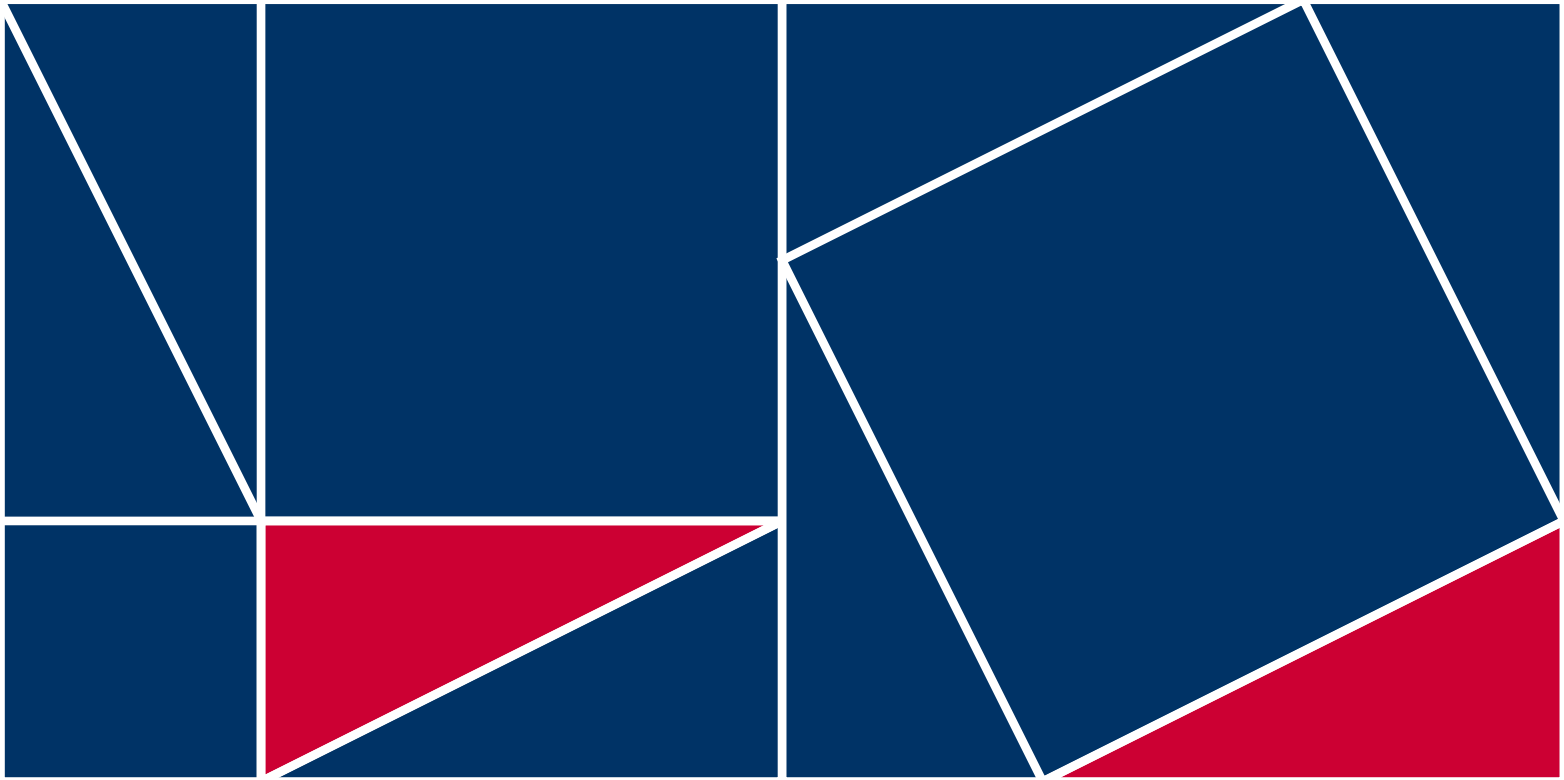
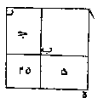


Al-Khwarismi, Hisab al-jabr w'al-muqabala, 830

علي تسعة ونلقيس ليتم المسطح الاعظم الذي هو سطح ره فيلزم
ذلك كله اربعة وستين فاخذنا جذرها وهو ثمانية وهو احد
اضلاع السطح الاعظم فاذا نقصنا منه مثل ما زدنا عليه وهو
خمسة بقي ثلثة وهو ضلع سطح اب الذي هو المال وهو جذره
والمال تسعة وهذه صورته



واما مال واحد وعشرون درهما يعدل عشرة اجذاره فانا
نجعل المال سطحا مربعا مجهول الاضلاع وهو سطح آد ثم نضم
اليه سطحا متوازي الاضلاع عرضه مثل احد اضلاع سطح آد وهو
ضلع وز والسطح وب فصار طول السطحين جميعا ضلع جه
وقد علمنا ان طوله عشرة من العدد لان كل سطح مربع
معادي الاضلاع والنزايا فان احد اضلاعه مضروبا في واحد جذر
ذلك السطح وفي اثنين جذراه فلما قال مال واحد وعشرون
يعدل عشرة اجذاره علمنا ان طول ضلع وج عشرة اعداد لان
ضلع جد جذر المال فقسمنا ضلع جه بنصفين علي نقطة

. . . what is the square which combined with ten of its roots will give a sum total of 39? The manner of solving this type of equation is to take one-half of the roots just mentioned. . . . Therefore take 5, which multiplied by itself gives 25, an amount which you add to 39 giving 64. Having taken then the square root of this which is 8, subtract from it half the roots, 5 leaving 3.

$$x^2 + 10x = 39$$

$$x^2 + 10x + 25 = 39 + 25 = 64$$

$$x + 5 = 8$$

$$x = 3$$

Algebraic manipulation

Problem

The expression

$$0.6\left(\frac{x+y+z}{3}\right)$$

is the contribution to a student's final score from three test scores. What is a different way of writing this? Which way should a student use in order to

- calculate the total test contribution to their final grade
- calculate the effect of getting 10 more points on test 2

Responses

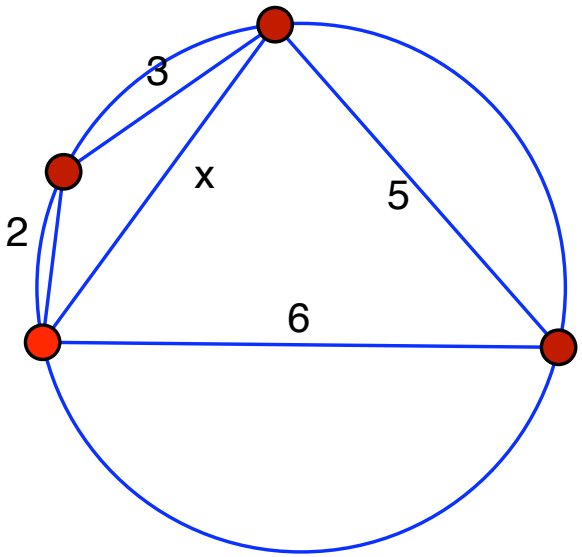
$$0.6\left(\frac{x+y+z}{3}\right),\ 0.2x+0.2y+0.2z,\ \frac{x}{5}+\frac{y}{5}+\frac{z}{5},\ \ldots$$

(1) $0.6\left(\frac{x+y+z}{3}\right)$ (2) $0.2x + 0.2y + 0.2z$

Student A: I wrote (2) because I thought that the original expression said the average of the 3 tests was worth 60%, so each test was worth 20%. But I'm not sure it is right.
Student B: (1) and (2) are obviously the same!
Student A: How you can see that just by looking at them?
Student B: You just move the 3 over so it's dividing the 0.6, which gives you 0.2, then distributed the 0.2.
Instructor: How do you know you can move the 3 over? What rule says you can do that?
Student B: Isn't it because you only have division and multiplication, so it's the commutative law?
Instructor: But division isn't commutative.
Student C: But you can write division as multiplication. Just write it as multiplication by 1/3.
Student A: Oh yeah! [Discussion shifts to associative law.]

Looking at expressions

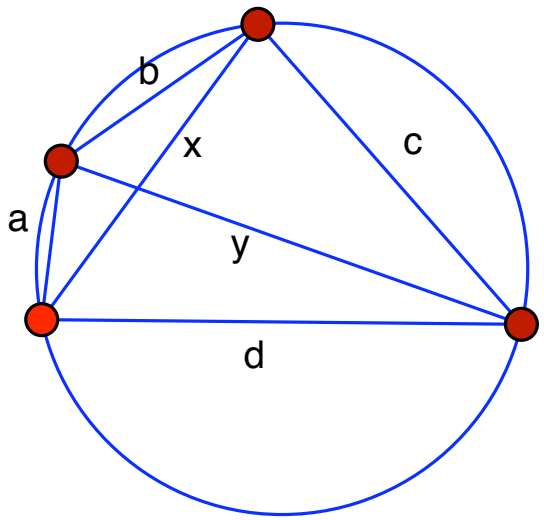
Given a cyclic quadrilateral whose sides are 2,3,5,6. Find the length of the square of the diagonal which makes a triangle with sides of length 2 and 3.



$$\begin{aligned} x^2 &= 4 + 9 - 2 \cdot 6 \cos\theta \\ &= 25 + 36 + 2 \cdot 30 \cos\theta \end{aligned}$$

Demo

Ptolemy's theorem



$$
\begin{aligned}
x^2 &= \frac{b^2cd + a^2cd + abc^2 + abd^2}{ab + cd} \\
&= \frac{bcbd + acad + acbc + adbd}{ab + cd} \\
&= \frac{(ac + bd)(ad + bc)}{ab + cd} \\
y^2 &= \frac{(bd + ca)(ba + cd)}{bc + da}
\end{aligned}
$$

Ptolemy's Theorem

$$xy = ac + bd$$